Half-linear differential equations, the Karamata theory, and the de Haan theory (Qualitative theory of ordinary differential equations in real domains)

AUTHOR(S):

Rehak, Pavel

数理解析研究所講究録
第 1993 巻 2016 年 1-25

Half-linear differential equations, the Karamata theory, and the de Haan theory

Pavel Řehák*

Institute of Mathematics, Academy of Sciences CR,
Žižkova 22, CZ-61662 Brno, Czech Republic,
rehak@math.cas.cz

1 Introduction

Let us consider the half-linear differential equation

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \tag{1}$$

where r, p are positive continuous functions on $[a, \infty)$ and $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$. We survey here recent results on asymptotic theory of equation (1) in the framework of the Karamata theory of regular variation and the de Haan theory, taken in particular from [29, 34, 35, 37]. We concentrate on establishing asymptotic formulae for all eventually positive solutions of (1) (which are necessarily eventually monotone).

Our results can be understood in several ways. We solve open problems posed in the literature ([31, 34, 37]). We generalize results for linear differential equations ([6, 10, 23, 24, 25, 31]); many of our observations are new even in the linear case. We provide a refinement on information about behavior of solutions of (1) in standard asymptotic classes ([2, 3]). We describe behavior of regularly and rapidly varying solutions which are known to exist ([3, 12, 13, 31, 34]). We prove regular or rapid variation of all positive solutions.

That the theory of regularly varying functions can be very useful in the study of asymptotic properties of differential equations has been shown in many works, see the important monograph [18] (which summarizes the research up to 2000) and the recent survey text [31]. Half-linear differential equations were studied in this framework in [3, 12, 13, 15, 16, 17, 26, 29, 31, 34, 35, 37].

*Supported by RVO 67985840.

The paper is organized as follows. In the next section we recall a several basic information about nonoscillatory solutions of (1). Some concepts from the Karamata theory of regularly varying functions and the de Haan theory are presented in Section 3. Solutions in the de Haan classes Γ and Γ^- are studied in Section 4. In Section 5 we deal with solutions in the de Haan class Π (which are in fact slowly varying) and derive asymptotic formulae for them. Under the same setting, in particular when $t^\alpha p(t)/r(t) \to 0$ as $t \to \infty$, in Section 6 we study non-slowly varying solutions. The case when $t^\alpha p(t)/r(t) \to C > 0$ as $t \to \infty$ is considered in Section 7; regular variation of positive solutions is shown and asymptotic formulae are obtained. The results of the previous sections are used in Section 8, which presents a classification of nonoscillatory solutions in the framework of regular variation. A certain generalization and a border-line case are discussed in Section 9. The paper is concluded with indicating some of the possible directions for the future research.

As usual, the relation $f(t) \sim g(t)$ (as $t \to \infty$) means $\lim_{t\to\infty} f(t)/g(t) = 1$ and $f(t) = o(g(t))$ (as $t \to \infty$) means $\lim_{t\to\infty} f(t)/g(t) = 0$.

2 Nonoscillatory solutions

It is known (see [3, Chapter 4]) that (1) with positive r, p is nonoscillatory, i.e., all its solutions are eventually of constant sign. Without loss of generality, we work just with positive solutions, i.e., with the class

$$\mathcal{S} = \{y : y(t) \text{ is a positive solution of (1) for large } t\}.$$

We wish to include our results into the framework of a standard classification of nonoscillatory solutions, which is recalled in what follows. Because of the sign conditions on the coefficients, all positive solutions of (1) are eventually monotone, therefore any such a solution belongs to one of the following disjoint classes:

$$\mathcal{IS} = \{y \in \mathcal{S} : y'(t) > 0 \text{ for large } t\}, \quad \mathcal{DS} = \{y \in \mathcal{S} : y'(t) < 0 \text{ for large } t\}.$$

It can be shown that both these classes are nonempty (see ([3, Lemma 4.1.2]). The classes $\mathcal{IS}, \mathcal{DS}$ can be divided into four mutually disjoint subclasses:

$$\mathcal{IS}_\infty = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = \infty\right\}, \quad \mathcal{IS}_B = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = b \in \mathbb{R}\right\},$$
$$\mathcal{DS}_B = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = b > 0\right\}, \quad \mathcal{DS}_0 = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = 0\right\}.$$

Define the so-called quasiderivative $y^{[1]}$ of $y \in \mathcal{S}$ by $y^{[1]} = r\Phi(y')$. We introduce the following convention

$$\mathcal{IS}_{u,v} = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = u, \quad \lim_{t\to\infty} y^{[1]}(t) = v\right\}$$
$$\mathcal{DS}_{u,v} = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = u, \quad \lim_{t\to\infty} y^{[1]}(t) = v\right\}.$$

For the subscripts of $\mathcal{IS}$ and $\mathcal{DS}$, by $u = B$ resp. $v = B$ we mean that the value of u resp. v is a real nonzero number. Using this convention we further distinguish the following types of solutions which form subclasses in $\mathcal{DS}_0, \mathcal{DS}_B, \mathcal{IS}_B$, and $\mathcal{IS}_\infty$ (we list only those ones that are not a-priori excluded):

$$\mathcal{DS}_{0,0}, \mathcal{DS}_{0,B}, \mathcal{DS}_{B,0}, \mathcal{DS}_{B,B}, \mathcal{IS}_{B,B}, \mathcal{IS}_{B,\infty}, \mathcal{IS}_{\infty,B}, \mathcal{IS}_{\infty,\infty}. \tag{2}$$

More information about (non)existence of solutions in these subclasses can be found in [2] and [3, Chapter 4].

No matter whether p is positive, if (1) is nonoscillatory, then there exists a nontrivial solution y of (1) such that for every nontrivial solution u of (1) with $u \neq \lambda y$, $\lambda \in \mathbb{R}$, we have $y'(t)/y(t) < x'(t)/x(t)$ for large t, see [3, Section 4.2], [22]. Such a solution is said to be a principal solution. Solutions of nonoscillatory equation (1) which are not principal, are called nonprincipal solutions. Principal solutions are unique up to a constant multiple. Denote $\mathfrak{P} = \{y \in \mathcal{S} : y \text{ is principal}\}$.

The so-called Riccati technique for (1) (in combination with other tools) plays an important role in some of the proofs. Its basic idea is as follows, see [3, Chapter 1]. Let $y \in \mathcal{S}$. Denoted $w = r\Phi(y'/y)$, it satisfies the generalized Riccati equation

$$w' - p(t) + (\alpha - 1)r^{1-\beta}(t)|w|^\beta = 0, \tag{3}$$

where β denotes the conjugate number of α, i.e., $1/\alpha + 1/\beta = 1$. Another substitution (introduced in [4]) $v = h^\alpha w - rh\Phi(h')$, $h \in C^1$, $h(t) \neq 0$ leads to a modified Riccati equation, and is very useful in the proof of Theorem 6 and Theorem 7. This substitution supplies — to some extent — the ("linear") transformation of dependent variable $y = hu$ in equation (1), which does not work in the half-linear case because of lack of additivity. Recall that another serious limitation in the theory of half-linear differential equations is the absence of a reduction of order formula; the reason is that there is no reasonable Wronskian identity for half-linear equations.

By Φ^{-1} we mean the inverse of Φ, i.e., $\Phi^{-1}(u) = |u|^{\beta-1}\operatorname{sgn} u$. If $\alpha = 2$, then $\Phi = \Phi^{-1} = \mathrm{id}$ and (1) reduces to the linear equation $(r(t)y')' = p(t)y$.

3 The Karamata theory of regular variation and the de Haan theory

In this section we recall some of the basic concepts of the Karamata theory of regularly varying functions and the de Haan theory, which are useful in studying differential equations. For more information on regular variation see the monographs [1, 7, 9].

A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index* ϑ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda > 0; \tag{4}$$

we write $f \in \mathcal{RV}(\vartheta)$. If $\vartheta = 0$, then we speak about *slowly varying* functions; we write $f \in \mathcal{SV}$, thus $\mathcal{SV} = \mathcal{RV}(0)$.

The so-called Uniform Convergence Theorem (see e.g. [1]) says that if $f \in \mathcal{RV}(\vartheta)$, then relation (4) holds uniformly on each compact λ-set in $(0, \infty)$.

It follows that $f \in \mathcal{RV}(\vartheta)$ if and only if there exists a function $L \in \mathcal{SV}$ such that $f(t) = t^{\vartheta} L(t)$ for every t. The slowly varying component of $f \in \mathcal{RV}(\vartheta)$ will be denoted by L_f, i.e.,

$$L_f(t) := \frac{f(t)}{t^{\vartheta}}.$$

The Representation Theorem (see e.g. [1]) says the following: $f \in \mathcal{RV}(\vartheta)$ if and only if

$$f(t) = \varphi(t) t^{\vartheta} \exp\left\{ \int_a^t \frac{\psi(s)}{s} \, \mathrm{d}s \right\}, \tag{5}$$

$t \geq a$, for some $a > 0$, where φ, ψ are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = 0$.

A function $f \in \mathcal{RV}(\vartheta)$ can alternatively be represented as

$$f(t) = \varphi(t) \exp\left\{ \int_a^t \frac{\omega(s)}{s} \, \mathrm{d}s \right\}, \tag{6}$$

$t \geq a$, for some $a > 0$, where φ, ω are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \omega(t) = \vartheta$.

From many interesting properties of $\mathcal{RV}$ functions, which are useful in our theory, let us mention here only the following Karamata Integration Theorem (see e.g. [1, 7]): Let $L \in \mathcal{SV}$. (i) If $\vartheta < -1$, then $\int_t^{\infty} s^{\vartheta} L(s) \, \mathrm{d}s \sim \frac{1}{-\vartheta-1} t^{\vartheta+1} L(t)$ as $t \to \infty$. (ii) If $\vartheta > -1$, then $\int_a^t s^{\vartheta} L(s) \, \mathrm{d}s \sim \frac{1}{\vartheta+1} t^{\vartheta+1} L(t)$ as

$t \to \infty$. (iii) If $\int_a^\infty L(s)/s\,\mathrm{d}s$ converges, then $\widetilde{L}(t) = \int_t^\infty L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; if $\int_a^\infty L(s)/s\,\mathrm{d}s$ diverges, then $\widetilde{L}(t) = \int_a^t L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; in both cases, $L(t)/\widetilde{L}(t) \to 0$ as $t \to \infty$.

A regularly varying function f is said to be *normalized regularly varying*, we write $f \in \mathcal{NRV}(\vartheta)$, if $\varphi(t) \equiv C$ in (5) or in (6). If (5) holds with $\vartheta = 0$ and $\varphi(t) \equiv C$, we say that f is *normalized slowly varying*, we write $f \in \mathcal{NSV}$.

The following type of extension of $\mathcal{RV}$ functions was introduced in [11]. Motivation was primarily for purposes of studying differential equations. Consider a continuously differentiable function ω which is positive and satisfies $\omega'(t) > 0$ for $t \in [b, \infty)$ and $\lim_{t\to\infty} \omega(t) = \infty$. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying of index ϑ with respect to ω* if $f \circ \omega^{-1} \in \mathcal{RV}(\vartheta)$; we write $f \in \mathcal{RV}_\omega(\vartheta)$. If $\vartheta = 0$, then f is called *slowly varying with respect to ω*; we write $f \in \mathcal{SV}_\omega$.

A function is called *rapidly varying*, we write $f \in \mathcal{RPV}(\infty)$ resp. $f \in \mathcal{RPV}(-\infty)$, if relation (4) holds with $\vartheta = \infty$ resp. $\vartheta = -\infty$; by λ^∞ we mean ∞ when $\lambda > 1$, and 0 when $\lambda \in (0,1)$, similarly for $\lambda^{-\infty}$. While $\mathcal{RV}$ functions of non-zero indices behave like power functions (up to a factor which varies "more slowly"), $\mathcal{RPV}$ functions have a behavior close to that of exponential functions.

We now recall useful subclasses of $\mathcal{SV}$ and $\mathcal{RPV}$ functions, namely the classes Pi and Gamma, which were studied by de Haan and others.

A measurable function $f : [a, \infty) \to \mathbb{R}$ is said to belong to the class Π if there exists a function $w : (0, \infty) \to (0, \infty)$ such that for $\lambda > 0$

$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{w(t)} = \ln \lambda; \tag{7}$$

we write $f \in \Pi$ or $f \in \Pi(w)$. The function w is called an *auxiliary function* for f. The class Π, after taking absolute values, forms a proper subclass of $\mathcal{SV}$.

We write $f \in \Pi\mathcal{RV}(\vartheta; w)$ if $t^{-\vartheta} f \in \Pi(w)$. Then we speak about Π-regular variation; this concept was introduced in [5].

A nondecreasing function $f : \mathbb{R} \to (0, \infty)$ is said to belong to the *class* Γ if there exists a function $v : \mathbb{R} \to (0, \infty)$ such that

$$\lim_{t\to\infty} \frac{f(t + \lambda v(t))}{f(t)} = e^\lambda \quad \text{for all } \lambda \in \mathbb{R};$$

we write $f \in \Gamma$ or $f \in \Gamma(v)$. The function v is called an *auxiliary function* for f. Further, $f \in \Gamma_-(v)$ if $1/f \in \Gamma(v)$. We have $\Gamma(v) \subset \mathcal{RPV}(\infty)$ and $\Gamma_-(v) \subset \mathcal{RPV}(-\infty)$.

A measurable function $f : \mathbb{R} \to (0, \infty)$ is called *Beurling slowly varying* if

$$\lim_{t\to\infty} \frac{f(t + \lambda f(t))}{f(t)} = 1 \quad \text{for all } \lambda \in \mathbb{R};$$

we write $f \in \mathcal{BSV}$. If the relation holds uniformly on compact λ-sets, then f is called *self-neglecting*; we write $f \in \mathcal{SN}$.

4 Solutions in the classes Γ and Γ_-

We start with increasing solutions. The proof of the next statement can be found in [37]; the important tools are the Riccati technique (which involves suitable weight functions), the theorem on differential inequalities, and the properties of Beurling slowly varying functions and self-neglecting functions. Note that this result was obtained in [29] in the special case when $r(t) \equiv 1$ and then extended in the same paper to the case $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$ via a suitable transformation of dependent variable. Using different techniques, in [37] we extended the theorem to the case when the integral can also converge. Moreover, in [37], we do not need to distinguish whether the integral $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s$ converges or diverges. Note that, in contrast to the linear case, the transformation of dependent variable (which can transform a 'convergent' case into a 'divergent' one) is not at disposal for equation (1).

Theorem 1. *Suppose that* $J_1 = \lim_{T\to\infty} \int_a^T r^{1-\beta}(t) \left(\int_a^t p(s)\mathrm{d}s\right)^{\beta-1} \mathrm{d}t = \infty$ *and* $\left(\frac{r}{p}\right)^{\frac{1}{\alpha}} \in \mathcal{BSV}$. *If there exists a function* $\tilde{f} \in C^1$ *satisfying*

$$\tilde{f}(t) \sim f(t) := \left(\frac{p(t)}{r(t)}\right)^{-\frac{1}{\beta}} \frac{1}{r(t)} \quad \textit{and} \quad \lim_{t\to\infty} \frac{\tilde{f}'(t)}{\tilde{f}^2(t)p(t)} = 0, \tag{8}$$

then $\mathcal{IS} = \mathcal{IS}_\infty \subset \Gamma\left(\left[\frac{(\alpha-1)r}{p}\right]^{\frac{1}{\alpha}}\right)$.

The statement of Theorem 1 can be reformulated in terms of generalized $\mathcal{RV}$ functions. Assume for simplicity that $r = 1$. Then $\mathcal{IS} \subset \mathcal{NRV}_\omega((\alpha - 1)^{-\frac{1}{\alpha}})$, where $\omega(t) = \exp\left\{\int_a^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right\}$, see [29].

A result in the spirit of Theorem 1 for the equation $y'' = p(t)y$ (i.e., for (1) with $\alpha = 2$ and $r(t) = 1$) can be found in [7, 23, 24, 25]. Note that the Wronskian identity which is used in the proof of the linear case is not at our disposal for equation (1).

Next we deal with decreasing solutions. Some of the steps in the proof are similar to those of the previous theorem, but some of them require a quite different approach, see [37]. Besides, the concept of principal solution plays an important role. As far as we know, the only result that deals with a solution in the class Γ_- was obtained in [24] for the linear equation, i.e. when $\alpha = 2$, in the special case when $r(t) \equiv 1$. Using a different method we extend the statement to a quite wider class of equations and, moreover, we deal with an entire subclass of decreasing solutions. In some aspects, the following result is new even in the linear case, see also the remarks presented after Corollary 1.

Theorem 2. *Suppose that* $J_2 = \lim_{T\to\infty} \int_a^T r^{1-\beta}(t) \left(\int_t^T p(s)\mathrm{d}s\right)^{\beta-1} \mathrm{d}t = \infty$ *or* $J_1 < \infty$, *and* $\left(\frac{r}{p}\right)^{\frac{1}{\alpha}} \in \mathcal{BSV}$. *If there exist functions* $\tilde{p}, \tilde{r} \in C^1$ *such that* $\tilde{f} := \left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\beta}} \frac{1}{\tilde{r}}$ *satisfies* (8), *then* $\mathcal{DS}_0 \subset \Gamma_-\left(\left[\frac{(\alpha-1)r}{p}\right]^{\frac{1}{\alpha}}\right)$.

The following corollary gives a precise classification of all solutions of (1) in terms of de Haan classes Γ and Γ_-.

Corollary 1. *Assume that* $J_1 = J_2 = \infty$ *and that* $\left(\frac{r}{p}\right)^{\frac{1}{\alpha}} \in \mathcal{BSV}$. *If there exist functions* $\tilde{p}, \tilde{r}$ *such that* $\tilde{p} \sim p, \tilde{r} \sim r$ *and* $\tilde{f} := \left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\beta}} \frac{1}{\tilde{r}}$ *satisfies* (8), *then*

$$\mathcal{IS} \subset \Gamma\left(\left[\frac{(\alpha-1)r}{p}\right]^{\frac{1}{\alpha}}\right) \quad \textit{and} \quad \mathcal{DS} \subset \Gamma_-\left(\left[\frac{(\alpha-1)r}{p}\right]^{\frac{1}{\alpha}}\right).$$

As a by-product of Corollary 1, we get conditions guaranteeing that

$$\mathcal{IS} = \mathcal{IS}_\infty \subset \mathcal{RPV}(\infty) \quad \text{and} \quad \mathcal{DS} = \mathcal{DS}_0 \subset \mathcal{RPV}(-\infty).$$

A closer examination of the proofs of Theorem 1 and Theorem 2 shows that under the conditions of Corollary 1 we have guaranteed the existence of solutions y_i of (1) such that

$$y_i'(t) \sim \pm \left(\frac{p(t)}{r(t)(\alpha-1)}\right)^{\frac{1}{\alpha}} y_i(t) \tag{9}$$

as $t \to \infty$, $i = 1, 2$. If $\alpha = 2$ and $r(t) = 1$, then (9) reduces to $y_i'(t) \sim \pm\sqrt{p(t)}y_i(t)$, y_i being solutions of $y'' = p(t)y$. The same formulas in the linear case were obtained in [10] by Hartman and Wintner under the assumptions

$\int_a^\infty \sqrt{p(s)}\,ds = \infty$ and $p'(t)/p^{\frac{3}{2}}(t) \to 0$ as $t \to \infty$. Omey in [23] rediscovered this statement for an increasing solution y_1 and showed that $y_1 \in \Gamma(p^{-\frac{1}{2}})$ under the assumption $p^{-\frac{1}{2}} \in \mathcal{BSV}$, see also [7, 24, 25]. A decreasing solution of $y'' = p(t)y$ which is in Γ_- was found by Omey in [24] via the reduction of order formula, with having at disposal an increasing solution in Γ. Note that this tool cannot be used in the half-linear case. We emphasize that in Corollary 1 we work with all (possible) decreasing solutions and with a general r, which makes this statement new also in the linear case.

The ideas of the proofs of Theorem 1 and Theorem 2 can be used to establish conditions — presented in the following theorem — which guarantee regular variation of solutions to (1), see [37].

Theorem 3. *Assume that there exist $\tilde{p}, \tilde{r} \in C^1$ such that*

$$\lim_{t\to\infty} \frac{\tilde{f}'(t)}{\tilde{f}^2(t)\tilde{p}(t)} = D \in \mathbb{R} \quad and \quad \lim_{t\to\infty}\left(\left(\frac{\tilde{p}(t)}{\tilde{r}(t)}\right)^{-\frac{1}{\alpha}}\right)' = C \in (0,\infty),$$

where $\tilde{f} = \left(\frac{\tilde{p}}{\tilde{r}}\right)^{-\frac{1}{\beta}}\frac{1}{\tilde{r}}$, and $\tilde{p}(t) \sim p(t), \tilde{r}(t) \sim r(t)$ as $t \to \infty$. Let $\varrho_1 > 0 > \varrho_2$ denote the roots of the equation

$$|\varrho|^\beta - \frac{D}{\alpha-1}\varrho - \frac{1}{\alpha-1} = 0.$$

(i) If $J_1 = \infty$, then $\mathcal{IS} \subset \mathcal{NRV}(\Phi^{-1}(\varrho_1)/C)$.
(ii) If $J_1 < \infty$ or $J_2 = \infty$, then $\mathcal{DS}_0 \subset \mathcal{NRV}(\Phi^{-1}(\varrho_2)/C)$.

In fact, regular variation of all solutions was obtained by a different way in the proof of the below presented Theorem 7, under the conditions $r \in \mathcal{NRV}(\gamma) \cap C^1$ and (25). If the coefficients r, p satisfy the conditions from both Theorems 3 and 7, then — as expected — the corresponding indices from both approaches coincide. There is also a relation with the results from [12, 13], see the last paragraph of Section 7 for more details.

5 Solutions in the class Π

In this section we deal with solutions in the class Π which are in fact slowly varying. Non-$\mathcal{SV}$ solutions under the same setting will be studied in the next section.

To simplify writing asymptotic formulae for solutions, we denote

$$\mathfrak{E}(\sigma,\tau,C,f) = \exp\left\{\int_\sigma^\tau (1+o(1))Cf(s)\,\mathrm{d}s\right\},$$

where $o(1)$ is meant either as $\tau \to \infty$ when $\tau < \infty$ or as $\sigma \to \infty$ when $\tau = \infty$.

The following conditions play an important role:

$$p \in \mathcal{RV}(\delta), \quad r \in \mathcal{RV}(\delta + \alpha), \tag{10}$$

$\delta \in \mathbb{R}$, and

$$\frac{L_p(t)}{L_r(t)} \to 0 \quad \text{as } t \to \infty. \tag{11}$$

Observe that (11) can be understood as $\lim_{t\to\infty} t^\alpha r(t)/p(t) = 0$. As shown in [37, Remark 11], the relation between the indices of regular variation of the coefficients r and p is quite natural when dealing with slowly varying solutions provided $\delta \neq -1$; a deeper discussion on this topic can be found in [34, 35]. Denote

$$G(t) = \left(\frac{tp(t)}{r(t)}\right)^{\beta-1}.$$

If (10) holds, then $G(t) = \frac{1}{t}\left(\frac{L_p(t)}{L_r(t)}\right)^{\beta-1} \in \mathcal{RV}(-1)$.

The proof of the following statement can be extracted from [34, 37]. In fact, there exist at least two approaches to the part in which we establish asymptotic formulae. One is based on properties of the class Π, the other one utilizes the Karamata theorem. Besides, we use properties of $\mathcal{RV}$ functions and various estimates.

Theorem 4. *Assume that* (10) *and* (11) *hold. If $\delta < -1$, then $\mathcal{DS} \subset \mathcal{NSV}$ and $-y(t) \in \Pi(-ty'(t))$ for any $y \in \mathcal{DS}$. If $\delta > -1$, then $\mathcal{IS} \subset \mathcal{NSV}$ and $y(t) \in \Pi(ty'(t))$ for any $y \in \mathcal{IS}$. Moreover, for any $y \in \mathcal{DS}$ when $\delta < -1$ and any $y \in \mathcal{IS}$ when $\delta > -1$ the following hold:*

(i) If $\int_a^\infty G(s)\,\mathrm{d}s = \infty$, then

$$y(t) = \mathfrak{E}\left(a, t, 1/\Phi^{-1}(\delta+1), G\right) \tag{12}$$

as $t \to \infty$, where $y \in \mathcal{DS}_{0,0}$ provided $y \in \mathcal{DS}$ and $\delta < -1$, while $y \in \mathcal{IS}_{\infty,\infty}$ provided $y \in \mathcal{IS}$ and $\delta > -1$.

(ii) If $\int_a^\infty G(s)\,\mathrm{d}s < \infty$, then

$$y(t) = N\mathfrak{E}\left(t, \infty, -1/\Phi^{-1}(\delta+1), G\right) \tag{13}$$

as $t \to \infty$, where $N := \lim_{t\to\infty} y(t) \in (0,\infty)$, where $y \in \mathcal{DS}_{B,0}$ provided $y \in \mathcal{DS}$ and $\delta < -1$, while $y \in \mathcal{IS}_{B,\infty}$ provided $y \in \mathcal{IS}$ and $\delta > -1$. Moreover, in any case, $|N - y| \in \mathcal{SV}$ and

$$\frac{L_p^{\beta-1}(t)}{L_r^{\beta-1}(t)(N - y(t))} = o(1) \tag{14}$$

as $t \to \infty$.

As a special case of Theorem 4 (when $\alpha = 2$, $r(t) = 1$, and considering decreasing solutions) we get [6, Theorem 0.1-A]. Other statements (including one of the approaches in the proof) are new also in the linear case.

Under condition (10), $\mathcal{SV}$ solutions of (1) necessarily decrease provided $\delta < -1$ while $\mathcal{SV}$ solutions of (1) necessarily increase provided $\delta > -1$, see [37, Remark 8, Remark 11].

6 Non-$\mathcal{SV}$ solutions when $t^\alpha p(t)/r(t) \to 0$

This section discusses a complementary case with respect to Theorem 4, namely we study increasing solutions when $\delta < -1$ and decreasing solutions when $\delta > -1$. Under the same setting (i.e., (10) and (11)) we can prove regular variation of these solutions where the index is equal to

$$\varrho := \frac{-1-\delta}{\alpha - 1},$$

and derive asymptotic formulas. Many of our considerations are new also in the linear case. Note that if $\delta = -\alpha$ (which happens, for instance, when $r(t) \equiv 1$ under condition (10)), then $\varrho = 1$.

Denote

$$H(t) = \frac{t^{\alpha-1}p(t)}{r(t)}.$$

If (10) holds, then $H(t) = \frac{1}{t} \cdot \frac{L_p(t)}{L_r(t)} \in \mathcal{RV}(-1)$. Since the convergence/divergence of the integrals $\int^\infty G$ and $\int^\infty H$ plays an important role, the following example is of interest.

Example 1. Taking r, p which satisfy (10) and such that $L_p(t)/L_r(t) = \ln^\gamma t$, $\gamma \in (-\infty, 0)$, we see that (11) is fulfilled. Moreover, if $\alpha < 2$ and $-1 < \gamma < 1-\alpha$ then $\int^\infty G < \infty$ while $\int^\infty H = \infty$, and if $\alpha > 2$ and $1 - \alpha < \gamma < -1$ then $\int^\infty G = \infty$ while $\int^\infty H < \infty$. Note that in the linear case (i.e., $\alpha = 2$), G and H are always the same.

There are more ways how to attack the above problem. The next theorem can actually be proved in two completely different ways, see [34]. The first one is based on the so-called reciprocity principle, with using Theorem 4. The second one is based on the Riccati technique in combination with various properties of regularly varying functions and functions in the class Π. Later we present a variant of that theorem, which is proved by a third way. Note that the reduction of order formula and fundamental set of solutions are another concepts which could be used to deal with the above problem for linear equations. However, these tools are not at disposal in the half-linear case.

Theorem 5. *Assume that* (10) *and* (11) *hold. If* $\delta < -1$, *then* $\mathcal{IS} \subset \mathcal{NRV}(\varrho)$. *If* $\delta > -1$, *then* $\mathcal{DS} \subset \mathcal{NRV}(\varrho)$. *Moreover, one has* $y^{[1]}(t) \in \Pi(tp(t)\Phi(y(t)))$ *for any* $y \in \mathcal{S} \cap \mathcal{NRV}(\varrho)$. *For any* $y \in \mathcal{IS}$ *when* $\delta < -1$ *and any* $y \in \mathcal{DS}$ *when* $\delta > -1$ *the following hold:*

(i) If $\int_a^\infty H(s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = A + \int_a^t r^{1-\beta}(s)\mathfrak{E}\left(a, s, (\beta-1)/\varrho^{\alpha-1}, H\right)\,\mathrm{d}s \tag{15}$$

as $t \to \infty$, *for some* $A \in \mathbb{R}$, *with* $y \in \mathcal{IS}_{\infty,\infty}$ *provided* $y \in \mathcal{IS}$ *and* $\delta < -1$, *while*

$$y(t) = \int_t^\infty r^{1-\beta}(s)\mathfrak{E}\left(a, s, -(\beta-1)/|\varrho|^{\alpha-1}, H\right)\,\mathrm{d}s \tag{16}$$

as $t \to \infty$, *with* $y \in \mathcal{DS}_{0,0}$ *provided* $y \in \mathcal{DS}$ *and* $\delta > -1$.

(ii) If $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = A + \int_a^t M^{\beta-1}r^{1-\beta}(s)\mathfrak{E}\left(s, \infty, -(\beta-1)/\varrho^{\alpha-1}, H\right)\,\mathrm{d}s \tag{17}$$

as $t \to \infty$, *for some* $A \in \mathbb{R}$, *with* $y \in \mathcal{IS}_{\infty,B}$ *provided* $y \in \mathcal{IS}$ *and* $\delta < -1$, *while*

$$y(t) = \int_t^\infty |M|^{\beta-1}r^{1-\beta}(s)\mathfrak{E}\left(s, \infty, (\beta-1)/|\varrho|^{\alpha-1}, H\right)\,\mathrm{d}s \tag{18}$$

as $t \to \infty$, *with* $y \in \mathcal{DS}_{0,B}$ *provided* $y \in \mathcal{DS}$ *and* $\delta > -1$, *where* $M = \lim_{t\to\infty} y^{[1]}(t) \in \mathbb{R} \setminus \{0\}$. *Moreover, in any case,* $M - y^{[1]} \in \mathcal{SV}$ *and*

$$\frac{L_p(t)}{L_r(t)(M - y^{[1]}(t))} = o(1) \tag{19}$$

as $t \to \infty$.

Example 2. Let $p(t) = t^\delta L_p(t)$, $\delta \neq -1$, with $L_p(t) = (\ln t)^{\nu_1} + g_1(t)$, and $r(t) = t^{\delta+\alpha}L_r(t)$ with $L_r(t) = (\ln t)^{\nu_2} + g_2(t)$, where $|g_i(t)| = o((\ln t)^{\nu_i})$ as $t \to \infty$, $i = 1, 2$, and $\nu_1 < \nu_2$. Then $L_p, L_r \in \mathcal{SV}$ and $\lim_{t\to\infty} L_p(t)/L_r(t) = 0$. For example, one can take $g_i(t) = \sin t$ or $g_i(t) = \ln(\ln t)$ provided $\nu_i > 0$. We have

$$G(t) \sim \frac{1}{t}(\ln t)^{\frac{\nu_1-\nu_2}{\alpha-1}} \quad \text{and} \quad H(t) \sim \frac{1}{t}(\ln t)^{\nu_1-\nu_2}$$

as $t \to \infty$. Having these relations, it is now easy to obtain formulae from Theorem 4 and Theorem 5 for solutions of the equation with the above defined coefficients.

Formulae (15) and (16) can be both written as

$$y(t) = tr^{1-\beta}(t)\mathfrak{E}\left(a, t, (\beta - 1)/\Phi(\varrho), H\right) \tag{20}$$

as $t \to \infty$, see [34].

If we assume — under the conditions of Theorem 7 — that y is a solution such that $\lim_{t\to\infty} y^{[1]}(t) = M \in \mathbb{R} \setminus \{0\}$, then the simple (and less precise) formula

$$y(t) = (1 + o(1))\frac{\Phi^{-1}(M)}{\varrho}tr^{1-\beta}(t) \tag{21}$$

as $t \to \infty$ can easily be obtained.

Assume that (10) holds. Condition (11) is necessary for the existence of $y \in \mathcal{S} \cap \mathcal{NSV}$ or $y \in \mathcal{S}_\cap\mathcal{NRV}(\varrho)$ provided $\delta \neq -1$, see [34, Remark 4] and [37, Remark 7, Remark 10].

In fact, to prove regular variation of our monotone solutions, the conditions (10) and (11), as well as the below discussed condition (25), can be somehow weakened into an integral form. Assuming regular variation of the coefficients of (1), conditions (11) and (25) are necessary for the existence of regularly varying solutions. See [34, 35, 37] for details.

The modified Riccati technique, which plays an important role in proving the results in the next section, in combination with Theorem 4 and Theorem 5, leads to the following variant of Theorem 5, see [35].

Theorem 6. *Let $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{NRV}(\delta + \alpha) \cap C^1$, $\delta \neq -1$, and (11) hold. Assume that $L \in \mathcal{SV}$, where*

$$L(t) = L_p(t)/L_r(t) + \Phi(\varrho)\left(\delta + \alpha - tr'(t)/r(t)\right). \tag{22}$$

If $\delta < -1$, then $\mathcal{IS} \subset \mathcal{NRV}(\varrho)$. If $\delta > -1$, then $\mathcal{DS} \subset \mathcal{NRV}(\varrho)$. Moreover, for any $y \in \mathcal{IS}$ when $\delta < -1$ and any $y \in \mathcal{DS}$ when $\delta > -1$, the following hold:

(i) If $\int_a^\infty L(s)/s\,\mathrm{d}s = \infty$, then

$$y(t) = t^\varrho\mathfrak{E}(a, t, 1/((\alpha - 1)(\Phi(\varrho))), L(s)/s) \tag{23}$$

as $t \to \infty$, where $y(t)/t^\varrho \nearrow \infty$ as $t \to \infty$ provided $y \in \mathcal{IS}$ and $\delta < -1$, and $y(t)/t^\varrho \searrow 0$ as $t \to \infty$ provided $y \in \mathcal{DS}$ and $\delta > -1$.

(ii) If $\int_a^\infty L(s)/s\,\mathrm{d}s < \infty$, then

$$y(t) = Dt^\varrho\mathfrak{E}(t, \infty, -1/((\alpha - 1)(\Phi(\varrho))), L(s)/s) \tag{24}$$

as $t \to \infty$, where $y(t)/t^\varrho \nearrow D = D(y) \in (0, \infty)$ as $t \to \infty$ provided $y \in \mathcal{IS}$ and $\delta < -1$, while $y(t)/t^\varrho \searrow D = D(y) \in (0, \infty)$ as $t \to \infty$ provided $y \in \mathcal{DS}$ and $\delta > -1$. Moreover, $|y(t)/t^\varrho - D| \in \mathcal{SV}$ and $L(t)/(D - y(t)/t^\varrho) = o(1)$ as $t \to \infty$.

If $r(t) = t^{\delta+\alpha}$ and $p(t) = t^\delta L_p(t)$, then L from Theorem 6 reduces to L_p.

7 $\mathcal{RV}$ solutions when $t^\alpha p(t)/r(t) \to C > 0$

As we have already noticed, (11) can be viewed as $\lim_{t\to\infty} t^\alpha p(t)/r(t) = 0$. A logical step is to assume that this limit is nonzero, i.e.,

$$\lim_{t\to\infty} t^\alpha p(t)/r(t) = C > 0. \tag{25}$$

Let λ_1, λ_2, $\lambda_1 > \lambda_2$, denote the (real) roots of the equation

$$|\lambda|^\beta + \frac{\gamma + 1 - \alpha}{\alpha - 1}\lambda - \frac{C}{\alpha - 1} = 0, \tag{26}$$

where $C > 0$, $\gamma \in \mathbb{R}$. We have $\lambda_2 < 0 < \lambda_1$. Denote $\vartheta_i = \Phi^{-1}(\lambda_i)$, $i = 1, 2$.

The following theorem was proved in [35]. Our method is new even in the linear case. By means of the Riccati technique — the approach is however different from the one used in Theorem 3 — we first show regular variation of all positive solutions. In the further step, an important role is played by the transformation into the modified Riccati equation. We can then apply Theorem 4. In fact, the latter transformation means — to some extent — a linearization of the problem.

Theorem 7. *Assume that $r \in \mathcal{NRV}(\gamma) \cap C^1$, condition (25) holds, and*

$$L_i(t) := t^\alpha p(t)/r(t) - C - \Phi(\vartheta_i)\,(\gamma - tr'(t)/r(t)) \in \mathcal{SV}, \tag{27}$$

$i = 1, 2$. Then $\mathcal{IS} \subset \mathcal{NRV}(\vartheta_1)$ and $\mathcal{DS} \subset \mathcal{NRV}(\vartheta_2)$. Moreover, for every $y_1 \in \mathcal{IS}$ and every $y_2 \in \mathcal{DS}$, $y_i(t) \in \Pi\mathcal{RV}(\vartheta_i; t^{1-\vartheta_i}y_i'(t) - \vartheta_i t^{-\vartheta_i} y_i(t))$, $i = 1, 2$, and the following hold:

(i) If, for $i = 1$ or $i = 2$, $\int_a^\infty L_i(s)/s\,\mathrm{d}s = \infty$, then

$$y_i(t) = t^{\vartheta_i}\mathfrak{E}(a, t, 1/((\alpha - 1)(\Phi(\vartheta_i) + C/\vartheta_i)), L_i(s)/s) \tag{28}$$

with $y_i(t)/t^{\vartheta_1} \nearrow \infty$ and $y_2(t)/t^{\vartheta_2} \searrow 0$ as $t \to \infty$.

(ii) If, for $i = 1$ or $i = 2$, $\int_a^\infty L_i(s)/s\,\mathrm{d}s < \infty$, then

$$y_i(t) = D_i t^{\vartheta_i}\mathfrak{E}(t, \infty, -1/((\alpha - 1)(\Phi(\vartheta_i) + C/\vartheta_i)), L_i(s)/s) \tag{29}$$

with $y_i(t)/t^{\vartheta_1} \nearrow D_1 = D_1(y_1) \in (0, \infty)$ and $y_2(t)/t^{\vartheta_2} \searrow D_2 = D_2(y_2) \in (0, \infty)$ as $t \to \infty$. Moreover, $|y_i(t)/t^{\vartheta_i} - D_i| \in \mathcal{SV}$ and

$$L_i(t)/(D_i - y_i(t)/t^{\vartheta_i}) = o(1) \tag{30}$$

as $t \to \infty$, $i = 1, 2$.

Next we give examples that present various types of situations as for behavior of L_1, L_2.

Example 3. (i) Let $r(t) = t^\gamma$, $\gamma \in \mathbb{R}$. Then $L_1(t) = t^\alpha p(t)/r(t) - C = L_2(t)$, and so the functions L_1, L_2 coincide.

(ii) Let $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$, and $p(t) = CL_r(t)t^{\gamma-\alpha}$, $C \in (0,\infty)$. Then $t^\alpha p(t)/r(t) = C$ and so

$$L_i(t) = -\Phi(\vartheta_i)\frac{tL_r'(t)}{L_r(t)}.$$

If, in addition, $L_r'(t) > 0$, then $L_1(t) < 0$ and $L_2(t) > 0$. Moreover, L_r can be found such that $L_2 \in \mathcal{SV}$. Indeed, if, for instance, $L_r(t) = \ln t$, then $tL_r'(t)/L_r(t) = 1/\ln t \in \mathcal{SV}$. Thus the situation where (25) is fulfilled and (27) holds for only one index can occur.

(iii) Let $r(t) = t^\gamma \ln t$, $\gamma \in \mathbb{R}$, and

$$p(t) = \left(C\ln t + \Phi(\vartheta_1) + \frac{C}{\ln t}\right),$$

$C \in (0,\infty)$. Then $t^\alpha p(t)/r(t) = C + \Phi(\vartheta_1)/\ln t + C/\ln^2 t \to C$ as $t \to \infty$, and $tr'(t)/r(t) = \gamma + 1/\ln t$. Further we have, $L_1(t) = C/\ln^2 t \in \mathcal{SV}$ and $L_2(t) = (\Phi(\vartheta_1) - \Phi(\vartheta_2))/\ln t + C/\ln^2 t \in \mathcal{SV}$. Since $(-1/\ln t)' = L_1(t)/C$, we get $\int_2^\infty L_1(s)/s\,\mathrm{d}s < \infty$. Moreover, $\int_2^\infty L_2(s)/s\,\mathrm{d}s = \infty$ because of $[\Phi(\vartheta_1) - \Phi(\vartheta_2)](\ln(\ln t))' = [\Phi(\vartheta_1) - \Phi(\vartheta_2)]/(t\ln t) < L_2(t)/t$. Thus we see that the case where $\int_a^\infty L_i(s)/s\,\mathrm{d}s = \infty$ while $\int_a^\infty L_{3-i}(s)/s\,\mathrm{d}s < \infty$ for one of $i \in \{1,2\}$ can generally occur in Theorem 7.

If $\alpha = 2$, $r(t) = 1$, and $i = 2$, then (28) reduces to the first formula in [6, Theorem 0.2]. The idea of the proof is different.

A fixed point approach was used in [12, 13] to derive conditions which guarantee the existence of a couple of $\mathcal{RV}$ solutions to equation (1) (with the indices which exactly correspond to the indices from Theorem 7). For $r(t) \not\equiv 1$, the concept of generalized regular variation was used. The conditions in [12, 13] are in a more general (integral) form than the ones involving $\lim_{t\to\infty} t^\alpha p(t)/r(t)$ (namely, for instance, that there exists the proper limit $\lim_{t\to\infty} t^\alpha \int_t^\infty p(s)\,\mathrm{d}s$ provided $r(t) \equiv 1$) and are shown to be necessary. On the other hand, here we guarantee regular variation of every eventually positive solution and establish asymptotic formulae for all of them. For the classical results on regular variation of solutions in the linear case see [18, 19].

8 Classification of nonoscillatory solutions in the framework of regular variation

Denote

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{S} \cap \mathcal{SV}, \quad \mathcal{S}_{\mathcal{RV}}(\vartheta) = \mathcal{S} \cap \mathcal{RV}(\vartheta),$$
$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{S} \cap \mathcal{NSV}, \quad \mathcal{S}_{\mathcal{NRV}}(\vartheta) = \mathcal{S} \cap \mathcal{NRV}(\vartheta).$$

If (10) holds, we set

$$J = \int_a^\infty \frac{1}{t}\left(\frac{L_p(t)}{L_r(t)}\right)^{\beta-1} \mathrm{d}t \quad \text{and} \quad R = \int_a^\infty \frac{1}{t} \cdot \frac{L_p(t)}{L_r(t)}\,\mathrm{d}t$$

We actually have

$$J = \int_a^\infty G(t)\,\mathrm{d}t \quad \text{and} \quad R = \int_a^\infty H(t)\,\mathrm{d}t$$

under condition (10). Further denote

$$\mathfrak{P} = \{y \in \mathcal{S} : y \text{ is principal}\}.$$

The results from Section 5 and Section 6 play an important role in proving the following corollary, see [34].

Corollary 2. *Let* (10) *and* (11) *hold.*
(i) Assume that $\delta < -1$.

(i-a) If $J = \infty$, *then* $\mathcal{S}_{\mathcal{NSV}} = \mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{0,0} = \mathfrak{P}$. *For any* $y \in \mathcal{DS}$ *formula* (12) *holds.*

(i-b) If $J < \infty$, *then* $\mathcal{S}_{\mathcal{NSV}} = \mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{B,0} = \mathfrak{P}$. *For any* $y \in \mathcal{DS}$ *formula* (13) *holds.*

(i-c) If $R = \infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}$. *For any* $y \in \mathcal{IS}$ *formulae* (15) *and* (20) *hold.*

(i-d) If $R < \infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty,B}$. *For any* $y \in \mathcal{IS}$ *formulae* (17) *and* (21) *hold.*

(ii) Assume that $\delta > -1$.

(ii-a) If $J = \infty$, *then* $\mathcal{S}_{\mathcal{NSV}} = \mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}$. *For any* $y \in \mathcal{IS}$ *formula* (12) *holds.*

(ii-b) If $J < \infty$, *then* $\mathcal{S}_{\mathcal{NSV}} = \mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{B,\infty}$. *For any* $y \in \mathcal{IS}$ *formula* (13) *holds.*

(ii-c) *If* $R = \infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0,0} = \mathfrak{P}$. *For any* $y \in \mathcal{DS}$ *formulae* (16) *and* (20) *hold.*

(ii-d) *If* $R < \infty$, *then* $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0,B} = \mathfrak{P}$. *For any* $y \in \mathcal{DS}$ *formulae* (18) *and* (21) *hold.*

Related to the setting of Section 7 is the following corollary, see [35].

Corollary 3. *Let* $r \in \mathcal{NRV}(\gamma) \cap C^1$, (25), *and* (27) *hold, where* $\lambda_1 = \Phi(\vartheta_1) > \lambda_2 = \Phi(\vartheta_2)$ *are the roots of* (26). *Then*

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{S}_{\mathcal{RV}}(\vartheta_1) = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}$$

and

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{S}_{\mathcal{RV}}(\vartheta_2) = \mathcal{DS} = \mathcal{DS}_{0,0} = \mathfrak{P}.$$

Moreover, for any $y_1 \in \mathcal{IS}$ *and* $y_2 \in \mathcal{DS}$ *respective formulae from Theorem 7 hold.*

As expected, the conditions from the previous corollaries naturally match the general existence conditions for (non–)emptiness of the standard asymptotic classes (established e.g. in [2, 3]) — this was revealed in [34]. Besides, we show a connection with regular variation and provide asymptotic formulae.

9 Generalization and the case $\delta = -1$

We now establish a generalization of the results from the previous Sections 5, 6, and 7. We distinguish the two cases, namely

$$\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty. \tag{31}$$

In the former case we denote $R(t) = \int_a^t r^{1-\beta}(s)\,\mathrm{d}s$ while in the latter one we denote $\overline{R}(t) = \int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s$. Further, R^{-1} stands for the inverse of R and Q^{-1} is the inverse of Q, where $Q = 1/\overline{R}$.

An important role in the proofs is played by certain transformations of independent variable, in combination with the results of Sections 5, 6, and 7, and various properties of $\mathcal{RV}$ functions.

Recall that if $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$, then a suitable choice in the transformation of dependent variable can transform (1) into $\frac{\mathrm{d}}{\mathrm{d}s}\left(r_T(s)\Phi\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\right) = p_T(s)\Phi(x)$, with $r_T = 1$, with preserving the type of the interval on which the equation is defined. This however is not the case when $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty$. A suitable transformation of dependent variable can transform the linear equation $(r(t)y')' = p(t)y$ with $\int_a^\infty 1/r(s)\,\mathrm{d}s < \infty$ into the equation $(\widehat{r}(t)x')' = \widehat{p}(t)y$ with $\int_a^\infty 1/\widehat{r}(s)\,\mathrm{d}s = \infty$. However, such a transformation is not at disposal for half-linear equation (1). That is why we examine separately the cases distinguished in (31).

As we will see, as a by-product, we solve the natural problem arising from Theorems 4 and 5, namely the missing case $\delta = -1$.

We start with a generalization of Theorem 4 (where $\delta < -1$) and Theorem 5 (where $\delta < -1$), see [35].

Theorem 8. *Assume that* $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$, $\widetilde{p} \in \mathcal{RV}(-\alpha)$, *where* $\widetilde{p} = (pr^{\beta-1}) \circ R^{-1}$, *and* $\lim_{t\to\infty} L_{\widetilde{p}}(t) = 0$. *Then for any* $y \in \mathcal{DS}$, *one has* $y \circ R^{-1} \in \mathcal{NSV}$ *and the following hold:*

(i-a) If $\int_a^\infty (R(s)p(s)r^{\beta-2}(s))^{\beta-1}\mathrm{d}s = \infty$, *then*

$$y(t) = \mathfrak{E}(a, t, -(1/(\alpha-1))^{\beta-1}, (Rpr^{\beta-2})^{\beta-1}) \tag{32}$$

as $t \to \infty$, *and* $y \in \mathcal{DS}_{0,0}$.

(i-b) If $\int_a^\infty (R(s)p(s)r^{\beta-2}(s))^{\beta-1}\mathrm{d}s < \infty$, *then*

$$y(t) = N\mathfrak{E}(t, \infty, (1/(\alpha-1))^{\beta-1}, (Rpr^{\beta-2})^{\beta-1}) \tag{33}$$

as $t \to \infty$, *with* $\lim_{t\to\infty} y(t) = N \in (0, \infty)$, $y \in \mathcal{DS}_{B,0}$, $y \circ R^{-1} - N \in \mathcal{SV}$, *and*

$$R^\alpha(t)p(t)r^{\beta-1}(t)/[\Phi(y(t) - N)] = o(1) \tag{34}$$

as $t \to \infty$.

For any $y \in \mathcal{IS}$, *one has* $y \circ R^{-1} \in \mathcal{NRV}(1)$ *and the following hold:*

(ii-a) If $\int_a^\infty R^{\alpha-1}(s)p(s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = R(t)\mathfrak{E}(a, t, \beta - 1, R^{\alpha-1}p) \tag{35}$$

as $t \to \infty$, *and* $y \in \mathcal{IS}_{\infty,\infty}$.

(ii-b) If $\int_a^\infty R^{\alpha-1}(s)p(s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = A + \int_a^t M^{\beta-1}r^{1-\beta}(s)\mathfrak{E}(s, \infty, 1-\beta, R^{\alpha-1}p)\mathrm{d}s \tag{36}$$

as $t \to \infty$, *for some* $A \in \mathbb{R}$, *with* $\lim_{t\to\infty} r(t)\Phi(y'(t)) = M \in (0, \infty)$, $y \in \mathcal{IS}_{\infty,B}$, $M - y^{[1]} \circ R^{-1} \in \mathcal{SV}$, *and* $R^\alpha(t)p(t)r^{\beta-1}(t)/(y^{[1]}(t) - M) = o(1)$ *as* $t \to \infty$.

The next result generalizes Theorem 4 (where $\delta > -1$) and Theorem 5 (where $\delta > -1$), see [35].

Theorem 9. *Assume that* $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty$, $\overline{p} \in \mathcal{RV}(\alpha-2)$, *where* $\overline{p} = (\overline{R}^2 p r^{\beta-1}) \circ Q^{-1}$, *and* $\lim_{t\to\infty} L_{\overline{p}}(t) = 0$. *Then for any* $y \in \mathcal{IS}$, *one has* $y \circ Q^{-1} \in \mathcal{NSV}$ *and the following hold:*

(i-a) If $\int_a^\infty (\overline{R}(s)p(s)r^{\beta-2}(s))^{\beta-1}\mathrm{d}s = \infty$, *then*

$$y(t) = \mathfrak{E}(a,t,(1/(\alpha-1))^{\beta-1},(\overline{R}pr^{\beta-2})^{\beta-1}) \tag{37}$$

as $t \to \infty$, *and* $y \in \mathcal{IS}_{\infty,\infty}$.

(i-b) If $\int_a^\infty (\overline{R}(s)p(s)r^{\beta-2}(s))^{\beta-1}\mathrm{d}s < \infty$, *then*

$$y(t) = N\mathfrak{E}(t,\infty,-(1/(\alpha-1))^{\beta-1},(\overline{R}pr^{\beta-2})^{\beta-1}) \tag{38}$$

as $t \to \infty$, *with* $\lim_{t\to\infty} y(t) = N \in (0,\infty)$, $y \in \mathcal{IS}_{B,\infty}$, $y \circ Q^{-1} - N \in \mathcal{SV}$, *and* $\overline{R}^\alpha(t)p(t)r^{\beta-1}(t)(\Phi(y(t)-N)) = o(1)$ *as* $t \to \infty$.

For any $y \in \mathcal{DS}$, *one has* $y \circ Q^{-1} \in \mathcal{NRV}(-1)$ *and the following hold:*

(ii-a) If $\int_a^\infty \overline{R}^{\alpha-1}(s)p(s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \overline{R}(t)\mathfrak{E}(a,t,\beta-1,\overline{R}^{\alpha-1}p) \tag{39}$$

as $t \to \infty$, *and* $y \in \mathcal{DS}_{0,0}$.

(ii-b) If $\int_a^\infty \overline{R}^{\alpha-1}(s)p(s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = \int_t^\infty |M|^{\beta-1}r^{1-\beta}(s)\mathfrak{E}(s,\infty,1-\beta,\overline{R}^{\alpha-1}p)\mathrm{d}s \tag{40}$$

as $t \to \infty$, *with* $\lim_{t\to\infty} r(t)\Phi(y'(t)) = M \in (-\infty,0)$, $y \in \mathcal{DS}_{0,B}$, $M - y^{[1]} \circ Q^{-1} \in \mathcal{SV}$, *and* $\overline{R}^\alpha(t)p(t)r^{\beta-1}(t)/(y^{[1]}(t)-M) = o(1)$ *as* $t \to \infty$.

Many expressions in Theorem 8 and Theorem 9 can be represented in terms of generalized regular variation. Indeed, in Theorem 8, instead of $\widetilde{p} \in \mathcal{RV}(-\alpha)$, we can write $pr^{\beta-1} \in \mathcal{RV}_R(-\alpha)$. Further, $y \in \mathcal{DS} \Rightarrow y \circ R^{-1} \in \mathcal{NSV}$ is in fact $\mathcal{DS} \subset \mathcal{NSV}_R$, and $y \in \mathcal{IS} \Rightarrow y \circ R^{-1} \in \mathcal{NRV}(1)$ is $\mathcal{IS} \subset \mathcal{NRV}_R(1)$. We also have $y - N \in \mathcal{SV}_R$ resp $M - y^{[1]} \in \mathcal{SV}_R$. In Theorem 9, instead of $\overline{p} \in \mathcal{RV}(\alpha-2)$, we can write $\overline{R}^2 pr^{\beta-1} \in \mathcal{RV}_Q(\alpha-2)$. Moreover, $\mathcal{DS} \subset \mathcal{NRV}_Q(-1)$ and $\mathcal{IS} \subset \mathcal{NSV}_Q$. We also have $y - N \in \mathcal{SV}_Q$ resp. $M - y^{[1]} \in \mathcal{SV}_Q$.

A similar generalization as for Theorems 4 and 5 can be made for Theorem 6, see [35].

A suitable transformation can give also a generalization of Theorem 7 which is presented next. For the proof see [35]. Let $\eta_1 > 0 > \eta_2$ denote the roots of $|\eta|^\beta - \eta - C/(\alpha-1) = 0$ and $\mu_i = \Phi^{-1}(\eta_i)$, $i = 1,2$.

Theorem 10. *(i) Let* $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$, $\lim_{t\to\infty} R^\alpha(t)p(t)r^{\beta-1}(t) = C \in (0,\infty)$, *and* $s^\alpha(pr^{\beta-1})\circ R^{-1}(s) - C \in \mathcal{SV}$. *Then* $\mathcal{IS} \subset \mathcal{NRV}_R(\mu_1)$ *and* $\mathcal{DS} \subset \mathcal{NRV}_R(\mu_2)$. *Moreover, for every* $y_1 \in \mathcal{IS}$ *and every* $y_2 \in \mathcal{DS}$, *the following hold:*

(i-a) If $\int_a^\infty Z(u)\,\mathrm{d}u = \infty$, *where* $Z(u) = R^{\alpha-1}(u)p(u) - Cr^{1-\beta}(u)/R(u)$, *then*

$$y_i(t) = \mathfrak{E}(a,t,1/[(\alpha-1)(\Phi(\mu_i) - C/\mu_i)], Z) \tag{41}$$

as $t \to \infty$, $i = 1,2$.

(i-b) If $\int_a^\infty Z(u)\,\mathrm{d}u < \infty$, *then*

$$y_i(t) = D_i R^{\mu_i}(t)\mathfrak{E}(t,\infty,-1/[(\alpha-1)(\Phi(\mu_i) - C/\mu_i)], Z) \tag{42}$$

as $t \to \infty$, *where* $\lim_{t\to\infty} y_i(t)/R^{\mu_i}(t) = D_i = D_i(y_i) \in (0,\infty)$, $i = 1,2$. *Moreover,* $|y_i/R^{\mu_i} - D_i| \in \mathcal{SV}_R$ *and* $[R^\alpha(t)p(t)r^{\beta-1}(t) - C]/[D_i - y_i(t)/R^{\mu_i}(t)] = o(1)$ *as* $t \to \infty$, $i = 1,2$.

(ii) Assume that $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty$, $\lim_{t\to\infty} \overline{R}^\alpha(t)p(t)r^{\beta-1}(t) = C \in (0,\infty)$, *and* $s^{-\alpha}(pr^{\beta-1})\circ Q^{-1}(s) - C \in \mathcal{SV}$. *Then* $\mathcal{IS} \subset \mathcal{NRV}_Q(-\mu_2)$ *and* $\mathcal{DS} \subset \mathcal{NRV}_Q(-\mu_1)$. *Moreover, for every* $y_1 \in \mathcal{IS}$ *and every* $y_2 \in \mathcal{DS}$, *the following hold:*

(ii-a) If $\int_a^\infty \overline{Z}(u)\,\mathrm{d}u = \infty$, *where* $\overline{Z}(u) = \overline{R}^{\alpha-1}(u)p(u) - Cr^{1-\beta}(u)/\overline{R}(u)$, *then*

$$y_i(t) = \overline{R}^{\mu_{3-i}}(t)\mathfrak{E}(a,t,1/[(\alpha-1)(-\Phi(\mu_{3-i}) + C/\mu_{3-i})], \overline{Z})$$

as $t \to \infty$, $i = 1,2$.

(ii-b) If $\int_a^\infty \overline{Z}(u)\,\mathrm{d}u < \infty$, *then*

$$y_i(t) = D_i\overline{R}^{\mu_{3-i}}(t)\mathfrak{E}(t,\infty,-1/[(\alpha-1)(-\Phi(\mu_{3-i}) + C/\mu_{3-i})], \overline{Z})$$

as $t \to \infty$, *where* $\lim_{t\to\infty} y_i(t)/R^{\mu_{3-i}}(t) = D_i = D_i(y_i) \in (0,\infty)$, $i = 1,2$. *Moreover,* $[\overline{R}^\alpha(t)p(t)r^{\beta-1}(t) - C]/[D_i - y_i(t)/R^{\mu_{3-i}}(t)] = o(1)$ *as* $t \to \infty$, *and* $|y_i/R^{\mu_{3-i}} - D_i| \in \mathcal{SV}_Q$, $i = 1,2$.

Next we show that the theorems in this section are indeed generalizations of theorems from the previous sections. For the proof of the lemma see [35].

Lemma 1. *Let* $r \in \mathcal{RV}(\delta+\alpha)$, $p \in \mathcal{RV}(\delta)$, $\delta \neq -1$, *and* $L_p(t)/L_r(t) \to 0$ *as* $t \to \infty$. *Then* $\widetilde{p} \in \mathcal{RV}(-\alpha)$ *and* $L_{\widetilde{p}}(s) \to 0$ *as* $s \to \infty$ *provided* $\delta < -1$ *resp.* $\overline{p} \in \mathcal{RV}(\alpha-2)$ *and* $L_{\overline{p}}(s) \to 0$ *as* $s \to \infty$ *provided* $\delta > -1$, *where* $\widetilde{p}$ *is defined in Theorem 8 and* $\overline{p}$ *is defined in Theorem 9.*

Thanks to the above lemma we see that if the assumptions of Theorems 4 and 5 with $\delta < -1$ resp. $\delta > -1$ are satisfied, then the assumptions of Theorem 8, resp. Theorem 9 are satisfied as well. As for the conclusions of these theorems, note that since — in the case $\delta < -1$ — one has $R^{-1} \in \mathcal{RV}(1/\varrho)$, we get that $y \in \mathcal{SV}$ provided $y \circ R^{-1} \in \mathcal{SV}$ and that $y \in \mathcal{RV}(\varrho)$ provided $y \circ R^{-1} \in \mathcal{RV}(1)$. Similarly we treat the case $\delta > -1$. Asymptotic formulae in Theorems 4 and 5 can be obtained from the general ones by applying the Karamata theorem to the expressions $R(t)$ and $\overline{R}(t)$.

Theorem 8 and Theorem 9 allow us to consider also the coefficients which are not regularly varying (for a slightly different approach to the non-$\mathcal{RV}$ coefficients case see [35]). Indeed, take, for example, $p(t) = e^{\gamma t}t^{\omega}$ and $r(t) = t^{\gamma}$ with $\gamma < 0, \omega < 0$. Then $\tilde{p}(s) = \left(\ln[(s+K)\gamma(1-\beta)]^{\frac{1}{\gamma(1-\beta)}}\right)^{\omega}[(s+K)\gamma(1-\beta)]^{-\alpha}$ for some $K \in (0,\infty)$, and $\int_a^{\infty} r^{1-\beta}(s)\,\mathrm{d}s = \infty$. Therefore, $\tilde{p} \in \mathcal{RV}(-\alpha)$ and $L_{\tilde{p}}(s) \to 0$ as $s \to \infty$.

Similarly we could examine the generalization of Theorem 7.

The generalization in the sense of Theorems 8 and 9 lies not only in allowing us to consider non-$\mathcal{RV}$ coefficients. In fact, they enable us to cover the missing case in Theorems 4 and 5, namely $\delta = -1$. For simplicity, take $r(t) = t^{\alpha-1}$. Then trivially $r \in \mathcal{RV}(\delta + \alpha)$, where $\delta = -1$. The condition for p which corresponds with this setting is $p \in \mathcal{RV}(-1)$. Clearly, $\int_1^{\infty} r^{1-\beta}(s)\,\mathrm{d}s = \infty$, thus we are in the situation of Theorem 8. Since $R(t) = \int_1^t r^{1-\beta}(s)\,\mathrm{d}s = \ln t$ and so $R^{-1}(s) = \exp s$, we have $\widetilde{p}(s) = e^s p(e^s) = L_p(e^s)$. Further, in view of $L_p(e^s) = s^{-\alpha}L_{\widetilde{p}}(s)$, we get $L_{\widetilde{p}}(\ln t) = \ln^{\alpha} t L_p(t)$. Now it is easy to see that the assumptions of Theorem 8 are fulfilled when

$$L_p \circ \exp \in \mathcal{RV}(-\alpha) \tag{43}$$

and

$$L_p(t)\ln^{\alpha} t \to 0 \quad \text{as } t \to \infty. \tag{44}$$

Under these conditions, $\mathcal{DS} \subset \mathcal{NSV}_{\ln}$ and $\mathcal{IS} \subset \mathcal{RV}_{\ln}(1)$. Moreover, if $y \in \mathcal{DS}$ and $\int_a^{\infty}(s^{2-\alpha}p(s)\ln s)^{\beta-1}\mathrm{d}s = \infty$, then

$$y(t) = \mathfrak{E}(a, t, -(1/(\alpha-1))^{\beta-1}, p(s)s^{2-\alpha}\ln s) \tag{45}$$

as $t \to \infty$. Deriving the particular formulae for the other cases via Theorem 8i s left to the reader.

Similar observations can be made also in the case $\int_a^{\infty} r^{1-\beta}(s)\,\mathrm{d}s < \infty$. This condition is fulfilled, for example, when $r(t) = t^{\alpha-1}\ln^{\gamma} t$, where $\gamma > \alpha - 1$. Then $\overline{R}(t) = \frac{(\ln t)^{1-\gamma(\beta-1)}}{\gamma(\beta-1)-1}$ and $Q^{-1}(s) = \exp\left\{\left(\frac{s}{\gamma(\beta-1)-1}\right)^{\frac{1}{\gamma(\beta-1)-1}}\right\}$.

Example 4. Let $r(t) = t^{\alpha-1}$, $p(t) = L_p(t)/t$ with $L_p(t) = \frac{1}{\ln^{\alpha} t}\cdot\frac{1}{(\ln_k t)^{\gamma}}$, $k \geq 2, \gamma > 0$, where $\ln_k t = \ln\ln_{k-1} t$. Since $(L_p \circ \exp)(t) = \frac{1}{t^{\alpha}}\cdot\frac{1}{(\ln_{k-1} t)^{\gamma}} \in \mathcal{RV}(-\alpha)$ and $L_p(t)\ln^{\alpha} t = \frac{1}{(\ln_k t)^{\gamma}} \to 0$ as $t \to \infty$, conditions (43) and (44) are fulfilled.

As for the integral in (45), if, for instance, $k = 2$, then we have (with $a = 2$)
$\int_2^\infty (s^{2-\alpha}p(s)\ln s)^{\beta-1}\mathrm{d}s = \int_2^\infty \frac{1}{s\ln s(\ln_2 s)^\gamma}\,\mathrm{d}s \begin{cases} < \infty & \text{for } \gamma > 1 \\ = \infty & \text{for } \gamma \in (0,1]. \end{cases}$

10 Further research

We have presented several methods for the study of asymptotic properties of linear and half-linear differential equations in the framework of regular variation. We believe that these ideas and their modifications will be useful also in other settings, for example:

- (Half-)linear equations of the form (1) with $p(t) < 0$ or with $p(t)$ which may change its sign. The first computations reveal, that for the case $p(t) < 0$, some of the steps work similarly as for $p(t) > 0$ — in particular, as for deriving asymptotic formulae. On the other hand, the structure of a solution space is completely different. Especially, the so-called intermediate solutions may cause difficulties when showing regular variation of all positive solutions.
- Nearly (half-)linear differential equations (i.e., the equations of the form (1) where Φ in both terms is replaced by a regularly varying function at infinity or at zero of index α). This is a quite delicate setting, and can be understood as an analysis of the borderline case (between sub-linearity and super-linearity) for a certain generalization of Emden-Fowler type equation. Some aspects of qualitative theory of the nearly-linear differential equation $(r(t)G(y'))' = p(t)F(y)$, where $F(|\cdot|), G(|\cdot|)$ are regularly varying at zero of index 1, are treated in [30]. It turns out that a modification of some methods known from the linear theory is a useful tool. However, as it was shown [30], some phenomena may occur which cannot happen in the linear case.
- (Half-)linear differential equations with deviated arguments; we expect that the theory of first order functional differential equations could be helpful when showing regular variation of all solutions and deriving asymptotic formulae. Existence of regularly varying solutions of linear and half-linear second order functional differential equations is studied e.g. in [14, 15].
- First order (half-)linear systems or higher order equations. Regular variations of solutions of n-the order linear differential equation is treated in [32]. It is of interest to obtain asymptotic formulae.
- (Half-)linear difference equations. Some results in this direction have already been obtained for linear equations, see [33]. For sufficient and nec-

essary conditions which guarantee regular or rapid variation of solutions of half-linear difference equations see [20, 21].

- (Half-)linear dynamic equations on time scales. It was shown in [38] that it is advantageous to distinguish two cases when studying regular variation on time scales, namely the graininess μ satisfies $\mu(t) = o(t)$ as $t \to \infty$ or $\mu(t) = (q-1)t$ with $q > 1$. The former case yields the theory similar to that of the classical continuous or discrete regular variation. The latter case leads to regular variation in q-calculus, which exhibits substantial differences from the classical continuous or discrete regular variation, see [27, 28]. Asymptotic formulae for solutions of linear q-difference equation (some of them being in the spirit of Sections 5, 6, 7, and 9) are established in [36].

Even though some results for linear or half-linear differential equations can be established via more approaches, not all these methods can be applicable in other settings. For instance, the reciprocity principle cannot be used in dynamic equations on time scales unless the graininess is somehow reasonable. Further, the facts like the absence of a chain rule (and, consequently, a substitution in the integral) in a discrete case or a time scale case might substantially affect availableness of some approaches.

Of course, there is also some space for improving the presented results. For instance: To examine whether some "purely linear" techniques (e.g. the use of the Wronskian identity or the transformation of dependent variable) can directly be applied to half-linear equation at least in an "asymptotic sense". To obtain asymptotic formulae under relaxation of some conditions, like $\lim_{t\to\infty} t^\alpha p(t)/r(t) = C$, into an integral form. See [8] for the linear case. Use the theory of regular variation and the de Haan theory to find more precise asymptotic formulas for solutions of (1) or estimations of remainders, see e.g. [6, Theorem 0.1-B] or [23, 25], respectively, for the linear case.

References

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular Variation, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, 1987.

[2] M. Cecchi, Z. Došlá, M. Marini, I. Vrkoč, Integral conditions for nonoscillation of second order nonlinear differential equations, Nonlinear Anal., Theory Methods Appl. 64 (2006), 1278–1289.

[3] O. Došlý, P. Řehák, Half-linear Differential Equations, Elsevier, North Holland, 2005.

[4] O. Došlý, M. Ünal, Half-linear differential equations: linearization technique and its application, J. Math. Anal. Appl. 335 (2007), 450–460.

[5] J. L. Geluk, Π-Regular Variation, Proc. Amer. Math. Soc. 82 (1981), 565–570

[6] J. L. Geluk, On slowly varying solutions of the linear second order differential equation. Proceedings of the Third Annual Meeting of the International Workshop in Analysis and its Applications. Publ. Inst. Math. (Beograd) (N.S.) 48 (1990), 52–60.

[7] J. L. Geluk, L. de Haan, Regular Variation, Extensions and Tauberian Theorems, CWI Tract 40, Amsterdam, 1987.

[8] J. L. Geluk, V. Marić, M. Tomić, On regularly varying solutions of second order linear differential equations, Differential Integral Equations 6 (1993), 329–336.

[9] L. de Haan, On Regular Variation and its Applications to the Weak Convergence of Sample Extremes, Mathematisch Centrum Amsterdam, 1970.

[10] P. Hartman, A. Wintner, Asymptotic integrations of linear differential equations, Amer. J. Math. 77 (1955), 45–86.

[11] J. Jaroš, T. Kusano, Self-adjoint differential equations and generalized Karamata functions, Bull. Cl. Sci. Math. Nat. Sci. Math. 29 (2004), 25–60.

[12] J. Jaroš, T. Kusano, T. Tanigawa, Nonoscillation theory for second order half-linear differential equations in the framework of regular variation, Results Math. 43 (2003), 129–149.

[13] J. Jaroš, T. Kusano, T. Tanigawa, Nonoscillatory half-linear differential equations and generalized Karamata functions, Nonlinear Anal. 64 (2006), 762–787.

[14] T. Kusano, V. Marić, On the existence and the asymptotic behaviour of regularly (slowly) varying solutions of some classes of second order functional differential equations with deviating argument, Int. J. Pure Appl. Math. 46 (2008), 231–236.

[15] T. Kusano, J. Manojlović, T. Tanigawa, Existence of regularly varying solutions with nonzero indices of half-linear differential equations with retarded arguments, Comput. Math. Appl. 59 (2010), 411–425.

[16] T. Kusano; V. Marić, Nonoscillatory linear and half-linear differential equations having regularly varying solutions, Adv. Math. Sci. Appl. 14 (2004), 351–357.

[17] T. Kusano, V. Marić, T. Tanigawa, Asymptotics of some classes of nonoscillatory solutions of second-order half-linear differential equations, Bull. Cl. Sci. Math. Nat. Sci. Math. 28 (2003), 61–74.

[18] V. Marić, Regular Variation and Differential Equations, Lecture Notes in Mathematics 1726, Springer-Verlag, Berlin-Heidelberg-New York, 2000.

[19] V. Marić, M. Tomić, A classification of solutions of second order linear differential equations by means of regularly varying functions. Proceedings of the Third Annual Meeting of the International Workshop in Analysis and its Applications, Publ. Inst. Math. (Beograd) (N.S.) 48 (1990), 199–207.

[20] S. Matucci, P. Řehák, Regularly varying sequences and second order difference equations, J. Difference Equ. Appl. 14 (2008), 17–30.

[21] S. Matucci, P. Řehák, Rapidly varying decreasing solutions of half-linear difference equations, Math. Comput. Modelling 49 (2009), 1692–1699.

[22] J. D. Mirzov, Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations, Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, 14. Masaryk University, Brno, 2004.

[23] E. Omey, Regular variation and its applications to second order linear differential equations, Bull. Soc. Math. Belg. Sér. B 33 (1981), 207–229.

[24] E. Omey, Rapidly varying behaviour of the solutions to a second order linear differential equation, Proc. of the 7th int. coll. on differential equations, Plovdiv, August 18-23, 1996. Utrecht: VSP (1997), 295–303

[25] E. Omey, On the class gamma and related classes of functions, retrieved April 19, 2013, from <www.edwardomey.com/nonsave/ OmeyBeurlingJuli2011Final.pdf>.

[26] Z. Pátíková, Asymptotic formulas for non-oscillatory solutions of perturbed Euler equation, Nonlinear Anal. 69 (2008), 3281–3290.

[27] P. Řehák, Second order linear q-difference equations: nonoscillation and asymptotics, Czechoslovak Mathematical Journal 61 (2011), 1107–1134.

[28] P. Řehák, On certain asymptotic class of solutions to second order linear q-difference equations, J. Phys. A: Math. Theor. 45 (2012) 055202.

[29] P. Řehák, De Haan type increasing solutions of half-linear differential equations, J. Math. Anal. Appl. 412 (2014), 236–243.

[30] P. Řehák, On decreasing solutions of second order nearly linear differential equations, Boundary Value Problems 62 (2014), 13 pp.

[31] P. Řehák, Nonlinear Differential Equations in the Framework of Regular Variation, AMathNet 2014, 207 pp. <users.math.cas.cz/~rehak/ndefrv>.

[32] P. Řehák, A few remarks on Poincaré-Perron solutions and regularly varying solutions, to appear in Math. Slov.

[33] P. Řehák, Asymptotic formulae for solutions of linear second-order difference equations, to appear in J. Difference Equ. Appl.

[34] P. Řehák, On some methods in half-linear asymptotic theory, submitted.

[35] P. Řehák, Asymptotic formulae for solutions of half-linear differential equations, submitted.

[36] P. Řehák, Asymptotic analysis of nonoscillatory solutions of q-difference equations via q-regular variation, submitted.

[37] P. Řehák, V. Taddei, Solutions of half-linear differential equations in the classes Gamma and Pi, to appear in Differ. Integral Equ.

[38] P. Řehák, J. Vítovec, Regular variation on measure chains, Nonlinear Anal. 72 (2010), 439–448.